Physics 513, Quantum Field Theory

Homework 2

Due Tuesday, 16th September 2003

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1. a) Studying classical field theory, we derived the Euler-Lagrange equations of motion,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0.$$

It is trivial to show that a field which is described by the Lagrangian given has the following equation of motion:

$$-m^2\phi - \frac{\partial V}{\partial \phi} - \partial_\mu \partial^\mu \phi = 0,$$
$$\implies \left(\partial_\mu \partial^\mu + m^2\right)\phi = -\frac{\partial V}{\partial \phi}. \tag{1.1}$$

Which is precisely the Klein-Gordon equation for a field in a potential V.

b) The canonical momentum is,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi. \tag{1.2}$$

Using π, we write the Hamiltonian for the field.

$$\begin{aligned} H = \int d^3x \mathcal{H} &= \int d^3x \, (\pi \partial_0 \phi - \mathcal{L}), \\ &= \int d^3x \left(\pi^2 - 1/2(\partial_0\phi)^2 + 1/2(\nabla\phi)^2 + 1/2 m^2\phi^2 + V(\phi)\right), \\ &= \frac{1}{2}\int d^3x \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2 + 2V(\phi)\right). \end{aligned} \tag{1.3}$$

c) With a complex scalar field, the Lagrangian becomes

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - V(\phi^*\phi).$$

Following the same procedure as in part (a) above, we use the Euler-Lagrange equation to show that

$$-m^2\phi^*\phi - \phi^* \frac{\partial V}{\partial \phi} - \phi \frac{\partial V}{\partial \phi^*} - \partial_\mu \phi^* \partial^\mu \phi = 0.$$
$$\implies \left(\partial_\mu \partial^\mu + m^2\right)\phi^*\phi = -\phi^* \frac{\partial V}{\partial \phi} - \phi \frac{\partial V}{\partial \phi^*} \tag{1.4}$$

It is relatively easy to show that canonical momenta of the field are

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^*;$$
$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \partial_0 \phi.$$

Using this expression for π, we will proceed as above to compute the Hamiltonian.

$$\begin{aligned} H = \int d^3x \mathcal{H} &= \int d^3x \, (\pi \partial_0 \phi - \mathcal{L}), \\ &= \int d^3x \left(\pi^*\pi - 1/2\pi^*\pi + 1/2\nabla\phi^*\nabla\phi + 1/2 m^2\phi^*\phi + V(\phi^*\phi)\right), \\ &= \frac{1}{2}\int d^3x \left(\pi^*\pi + \nabla\phi^*\nabla\phi + m^2\phi^*\phi + 2V(\phi^*\phi)\right). \end{aligned} \tag{1.5}$$

d) Let us derive the Noether current generated by a *global* phase rotation $\phi \to \phi' = e^{i\alpha}\phi$. It is clear that $\mathcal{L}' = \mathcal{L}$ because only modulus terms of ϕ appear in $\mathcal{L}$. We rewrite the global phase rotation to the first order as

$$\phi \to \phi' = e^{i\alpha}\phi \approx (1+i\alpha)\phi \Rightarrow \Delta\phi = i\phi;$$
$$\phi^* \to \phi'^* = e^{-i\alpha}\phi^* \approx (1-i\alpha)\phi^* \Rightarrow \Delta\phi^* = -i\phi^*. \tag{1.6}$$

We showed in class that the conserved Noether current associated with a symmetry is specified by

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^*, \\ &= \left(i\phi\partial^\mu\phi^* - i\phi^*\partial^\mu\phi\right), \\ &= i\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right). \end{aligned} \tag{1.7}$$

2. **a)** The Lagrangian for a source-free electromagnetic field is specified by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.1}$$

It is clear that $F_{\mu\nu}$ is antisymmetric, $F_{\mu\nu} = -F_{\nu\mu}$. From our knowledge of the metric tensor in Minkowski space, it is also clear that $F_{\mu\nu} = -F^{\mu\nu}$ if either μ or ν is zero and $F_{\mu\nu} = F^{\mu\nu}$ if both μ and ν are nonzero. Because the field strength tensor is antisymmetric, our calculation will be much easier.

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\left(F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{12}F^{12} + F_{13}F^{13} + F_{23}F^{23}\right), \\ &= \frac{1}{2}\left(F_{01}^2 + F_{02}^2 + F_{03}^2 - F_{12}^2 - F_{13}^2 - F_{23}^2\right), \\ &= \frac{1}{2}[(\partial_0 A_1 - \partial_1 A_0)^2 + (\partial_0 A_2 - \partial_2 A_0)^2 + (\partial_0 A_3 - \partial_3 A_0)^2 \\ &\quad - (\partial_1 A_2 - \partial_2 A_1)^2 - (\partial_1 A_3 - \partial_3 A_1)^2 - (\partial_2 A_3 - \partial_3 A_2)^2], \\ &= \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right). \end{aligned}$$

Now, let us try to find the Euler-Lagrange equations for motion for this field. Note that from our work above if it clear that,

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = 0.$$

After a short while of staring at the above equations, you should see that

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} &= \begin{cases} (\partial_\mu A_\nu - \partial_\nu A_\mu) & \text{if } \mu = 0 \text{ or } \nu = 0, \\ -(\partial_\mu A_\nu - \partial_\nu A_\mu) & \text{if } \mu,\nu \neq 0, \end{cases} \\ &= -F^{\mu\nu} = F^{\nu\mu}. \end{aligned}$$

So the equations of motion are simply

$$\partial_\mu F^{\nu\mu} = 0. \tag{2.2}$$

Knowing that $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = F^{ji}$, we can rewrite (2.2) as

$$\begin{aligned} \partial_\mu F^{0\mu} = \partial_i F^{0i} = 0 = -\partial_1 E^1 - \partial_2 E^2 - \partial_3 E^3 = 0, \\ \therefore \nabla\cdot\mathbf{E} = 0. \end{aligned} \tag{2.3}$$

The other equations also can be reduced to familiarity. Specifically,

$$\begin{aligned} \partial_\mu F^{\nu\mu} &= \partial_\mu F^{k\mu} = 0, \\ \Longrightarrow \partial_0 F^{k0} &= \partial_i F^{ki} = \epsilon^{ijk}\partial_i B_j, \\ \therefore \nabla\times\mathbf{B} &= \partial_0\mathbf{E}. \end{aligned} \tag{2.4}$$

These two equations represent half of Maxwell's equations for a source-free field. The other two equations relate the vector potential A_ν with the $\mathbf{E}$ and $\mathbf{B}$ fields. These two other equations were 'given.' We needed to know that $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = -\partial_0\mathbf{A} - \nabla A_0$ to write down the components of $\mathbf{E}$ and $\mathbf{B}$ in terms of $F_{\mu\nu}$.

b) We construct the energy-momentum tensor, $T^{\mu\nu}$, (using the equation derived in my unpublished QFT notes),

$$T^{\mu}_{\ \nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \mathcal{L}\delta^{\mu}_{\ \nu}, \tag{2.5}$$

It should be clear that by simply applying our results of part (a)

$$T^{\mu\nu} = F^{\lambda\mu}\partial^\nu A_\lambda - \mathcal{L}\delta^{\mu}_{\ \nu}.$$

This is not symmetric in μ and ν. Remember that the important aspect of $T^{\mu\nu}$ is that it is *conserved*, i.e. $\partial_\mu T^{\mu\nu} = 0$. To make $T^{\mu\nu}$ easier to work with, consider changing it to

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}.$$

Where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. By this antisymmetry, it is clear that

$$\partial_\mu \hat{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu\partial_\lambda K^{\lambda\mu\nu} = 0.$$

So $\hat{T}^{\mu\nu}$ is a conserved quantity for any $K^{\lambda\mu\nu}$ that is antisymmetric in its first two indices. Let $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$ which is certainly antisymmetric in λ and μ because of $F^{\mu\lambda}$. This allows us to rewrite $\hat{T}^{\mu\nu}$ in a much simpler form. (Note the use of the Euler-Lagrange equations to simplify line 2 below).

$$\begin{aligned}
\hat{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\lambda F^{\mu\lambda}A^\nu, \\
&= T^{\mu\nu} + A^\nu(\partial_\lambda F^{\mu\lambda}) + F^{\mu\lambda}(\partial_\lambda A^\nu), \\
&= T^{\mu\nu} + F^{\mu\lambda}(\partial_\lambda A^\nu), \\
&= F^{\lambda\mu}\partial^\nu A_\lambda + F^{\mu\lambda}\partial_\lambda A^\nu - \mathcal{L}\delta^{\mu}_{\ \nu}, \\
&= F^{\lambda\mu}(\partial^\nu A_\lambda - \partial_\lambda A^\nu) - \mathcal{L}\delta^{\mu}_{\ \nu}.
\end{aligned}$$

It should be clear that $\hat{T}^{\mu\nu} = \hat{T}^{\nu\mu}$. Now we are ready to derive the Hamiltonian and total momentum from $\hat{T}^{\mu\nu}$. First, the Hamiltonian is

$$\begin{aligned}
\mathcal{H} = \mathcal{E} = \hat{T}^{00}&, \\
&= E^i(\partial_i A^0 - \partial^0 A_i) - \mathcal{L}, \\
&= \mathbf{E}^2 - E^i\partial^0 A_i - \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \\
&= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2).
\end{aligned} \tag{2.6}$$

Note that in the last line of the derivation we had to set $E^i\partial^0 A_i = 0$. The total momentum of the field is

$$\begin{aligned}
S^k = T^{0k} &= -E^i(\partial^i A^k - \partial^k A^i), \\
&= E_i(\partial^i A^k - \partial^k A^i), \\
&= E_i\epsilon^{ijk}B_k, \\
\therefore \mathbf{S} &= \mathbf{E}\times\mathbf{B}.
\end{aligned} \tag{2.7}$$

3. a) The inner product, (f,g), will be defined

$$(f,g) \equiv i\int d^3x f^*(x)\partial_0 g(x) - g(x)\partial_0 f^*(x),$$

We show that (f,g) is independent of time. This is demonstrated by direct computation.

$$\begin{aligned}\partial_0(f,g) &= i\int d^3x\partial_0\left[f^*(x)\partial_0 g(x) - g(x)\partial_0 f^*(x)\right],\\ &= i\int d^3x\left[\partial_0 f^*(x)\partial_0 g(x) + f^*(x)\partial_0^2 g(x) - g(x)\partial_0^2 f^*(x) - \partial_0 f^*(x)\partial_0 g(x)\right],\\ &= i\int d^3x\left[f^*(x)\partial_0^2 g(x) - g(x)\partial_0^2 f^*(x)\right].\end{aligned}$$

Using the Klein-Gordon equation, this reduces to

$$\begin{aligned}\partial_0(f,g) &= i\int d^3x f^*(\nabla^2 - m^2)g - g(\nabla^2 - m^2)f^*,\\ &= i\int d^3x f^*\nabla^2 g - g\nabla f^*.\end{aligned}$$

We use Green's Theorem to reduce the equation above to

$$\partial_0(f,g) = i\int_S (f^*\nabla g - g\nabla f^*)\vec{n}\cdot da = 0. \tag{3.1}$$

The integral vanishes because we may assume that the fields go to zero at infinity.

b) Recall that the inverse Fourier transform of a Fourier transform of a function is the function itself.

$$f(k) = \int d^3x\left[e^{ikx}\int\frac{d^3k}{(2\pi)^3}e^{-ikx}f(k)\right].$$

Note that when we will express $\phi(x)$ in terms of ladder operators below, ϕ will be a function of the 4-vectors k and x. There is a minus sign to keep track of that is different from the book's 3-vector representation.

$$\phi(x) = \int\frac{d^3k}{(2\pi)^3}\frac{a}{\sqrt{2E_k}}\left(a_k e^{-ikx} + a_k^\dagger e^{ikx}\right).$$

We are now ready to derive the required identity. It will proceed by direct calculation.

$$\begin{aligned}a_k = (f_k(x),\phi(x)) &= i\int d^3x(f^*\partial_0\phi - \phi\partial_0 f^*),\\ &= i\int d^3x\left[e^{ikx}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\left(-iE_k a_k e^{-ikx} + iE_k a_k^\dagger e^{ikx}\right)\right.\\ &\qquad\left. - e^{ikx}\int\frac{d^3k}{(2\pi)^3}\frac{iE_k}{2E_k}\left(a_k e^{-ikx} + a_k^\dagger e^{ikx}\right)\right],\\ &= \int d^3x e^{ikx}\left[\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}\left(a_k e^{-ikx} - a_k^\dagger e^{ikx} + a_k e^{-ikx} + a_k^\dagger e^{ikx}\right)\right],\\ &= \int d^3x e^{ikx}\int\frac{d^3k}{(2\pi)^3}e^{-ikx}a_k = a_k,\\ &\therefore a_k = (f_k(x),\phi(x)) = a_k.\end{aligned} \tag{3.2}$$

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c) Let us derive the the commutation relation $\left[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. To find this commutation relation, we will first consider the fields in terms of ladder operators.

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger})e^{i\mathbf{p}\cdot\mathbf{x}};$$
$$\pi(\mathbf{y}) = \int \frac{d^3p'}{(2\pi)^3}\ (-i)\sqrt{\frac{\omega_{\mathbf{p}'}}{2}}(a_{\mathbf{p}'} - a_{-\mathbf{p}'}^{\dagger})e^{i\mathbf{p}'\cdot\mathbf{y}}.$$

Note that because the **p**'s are dummy variables, we cannot assume they are the same when we "mix" the integration, so we have called one **p**'.

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$$
$$= \int \frac{d^3pd^3p'}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\frac{-i}{2}\left(a_{\mathbf{p}}a_{\mathbf{p}'} - a_{\mathbf{p}}a_{-\mathbf{p}'}^{\dagger} + a_{-\mathbf{p}}^{\dagger}a_{\mathbf{p}'} - a_{-\mathbf{p}}^{\dagger}a_{-\mathbf{p}'}^{\dagger} - a_{\mathbf{p}'}a_{\mathbf{p}} - a_{\mathbf{p}'}a_{-\mathbf{p}}^{\dagger} + a_{-\mathbf{p}'}^{\dagger}a_{\mathbf{p}} + a_{-\mathbf{p}'}^{\dagger}a_{-\mathbf{p}}^{\dagger}\right)e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})}$$
$$= \int \frac{d^3pd^3p'}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\frac{i}{2}\left(a_{\mathbf{p}}a_{-\mathbf{p}'}^{\dagger} - a_{-\mathbf{p}'}^{\dagger}a_{\mathbf{p}} + a_{\mathbf{p}'}a_{-\mathbf{p}}^{\dagger} - a_{-\mathbf{p}}^{\dagger}a_{\mathbf{p}'}\right)e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})}\text{(cancelling like terms by symmetry)}$$
$$= \int \frac{d^3pd^3p'}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\frac{i}{2}\left(\left[a_{\mathbf{p}}, a_{-\mathbf{p}'}^{\dagger}\right] + \left[a_{\mathbf{p}'}, a_{-\mathbf{p}}^{\dagger}\right]\right)e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})}\quad\left(\text{note that }\left[a_{\mathbf{p}}, a_{-\mathbf{p}'}^{\dagger}\right] = \left[a_{\mathbf{p}'}, a_{-\mathbf{p}}^{\dagger}\right]\right)$$
$$= \int \frac{d^3pd^3p'}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}i\left[a_{\mathbf{p}}, a_{-\mathbf{p}'}^{\dagger}\right]e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})} = i\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{3.3}$$

Note that by the properties of the Dirac δ functional,

$$\int \frac{d^3pd^3p'}{(2\pi)^3}ie^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{y})} = i\delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

Applying this knowledge to (3.3) from above, $\left[a_{\mathbf{p}}, a_{-\mathbf{p}'}^{\dagger}\right]$ must satisfy

$$\int \frac{d^3pd^3p'}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}[a_{\mathbf{p}}, a_{-\mathbf{p}'}^{\dagger}] = 1.$$

This is identically satisfied if and only if we have that

$$\left[a_{\mathbf{p}}, a_{-\mathbf{p}'}^{\dagger}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{p}').$$

You can check this statement by noticing that this implies

$$\int \frac{d^3pd^3p'}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\left[a_{\mathbf{p}}, a_{-\mathbf{p}'}^{\dagger}\right] = \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{p}}}} = 1.$$

Therefore, noting our use of $-\mathbf{p}$, we may conclude that

$$\left[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}') \tag{3.4}$$

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